

CATEGORICAL RESOLUTIONS OF A CLASS OF DERIVED CATEGORIES

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ABSTRACT. By using the relative derived categories, we prove that if an Artin algebra A has a module T with $\text{inj.dim} T < \infty$ such that ${}^{\perp}T$ is finite, then the bounded derived category $D^b(A\text{-mod})$ admits a categorical resolution in the sense of [Kuz], and a categorical desingularization in the sense of [BO]. For CM-finite Gorenstein algebra, such a categorical resolution is weakly crepant. The similar results hold also for $D^b(A\text{-Mod})$.

Key words: (weakly crepant) categorical resolution, derived category, relative derived category, Gorenstein-projective object, CM-finite algebra

1. Introduction

1.1. A categorical resolution of an algebraic variety comes from looking for a minimal resolution of singularities. The functor $D^b(\tilde{X}) \rightarrow D^b(X)$ induced by a resolution of singular variety X enjoys some remarkable properties. This motivates the study of a categorical resolution of a triangulated category. A. Bondal and D. Orlov [BO, Section 5] define a *categorical desingularization of triangulated category* \mathcal{D} to be a pair $(D^b(\mathcal{A}), \mathcal{K})$, where \mathcal{A} is an abelian category of finite homological dimension, and \mathcal{K} a thick subcategory, such that $D \cong D^b(\mathcal{A})/\mathcal{K}$. A. Kuznetsov [Kuz] defines a *categorical resolution* of \mathcal{D} to be a triple $(\tilde{\mathcal{D}}, \pi_*, \pi^*)$, where $\tilde{\mathcal{D}}$ is an admissible subcategory of $D^b(\tilde{X})$ with \tilde{X} a smooth variety, $\pi_* : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ and $\pi^* : \mathcal{D}_{\text{perf}} \rightarrow \tilde{\mathcal{D}}$ are triangle functors satisfying (ii) and (iii) in Definition 2.3 below. If π^* is right adjoint to π_* , then it is called weakly crepant. M. Van den Bergh [Van] defines a non-commutative crepant resolution, this induces a categorical desingularization and a weakly crepant categorical resolution. For some of the other influential works in this area we refer to [Ab], [BKR], [BLV], [Kal], [Lun], and [SV].

1.2. In this paper we combine Kuznetsov's definition with Bondal-Orlov's one. See Definition 2.3. The reasons are: this is the case for a proper birational resolution of an algebraic variety of rational singularity, it contains more information, and applies to our purpose.

Two comments on Definition 2.3 are in order. First, we usually need to explicitly determine the perfect subcategory $\mathcal{D}_{\text{perf}}$ for a work. For an abelian category \mathcal{A} with enough projective objects, we have $D_{\text{perf}}^b(\mathcal{A}) := D^b(\mathcal{A})_{\text{perf}} \supseteq K^b(\mathcal{P})$, and $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$ in many important cases, where \mathcal{P} is the full subcategory of \mathcal{A} consisting of projective

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objects. For examples, if \mathcal{A} is the module category of a ring, or the finitely generated module category of an Artin algebra. Propositions 3.2 and 3.3 below say that this is also the case if \mathcal{A} is finitely filtrated, or \mathcal{A} is cocomplete.

Second, there are several ways for defining the smoothness of a triangulated category ([BO], [KS], [Kuz], [Lun], and [TV]). In this paper a triangulated category is smooth, if it is triangle-equivalent to $D^b(\mathcal{A})$ with \mathcal{A} an abelian category such that $D_{sg}^b(\mathcal{A}) := D^b(\mathcal{A})/D_{\text{perf}}^b(\mathcal{A}) = 0$ (see Definition 2.2). This singularity category $D_{sg}^b(\mathcal{A})$ is in the sense of R.O.Buchweitz [Buch] and D.Orlov [O2]. It is invariant under triangle-equivalences. An algebraic variety X is smooth if and only if $D_{sg}^b(X) = 0$ ([O2]). Definition 2.2 is almost same with the one in [BO, Section 5]. The reason for a replacement of finite homological dimension by $D_{sg}^b(\mathcal{A}) = 0$ is that in general the invariance of finite homological dimension under derived equivalences seems to be not known, although this is true in many important cases. Definition 2.2 is also a slight change of [Kuz, Definition 3.1], since in this paper we do not need to consider admissible subcategories. We hope that such a small change will not affect the minimality.

1.3. Given a full subcategory \mathcal{C} of an abelian category \mathcal{A} , the \mathcal{C} -relative derived category $D_{\mathcal{C}}^b(\mathcal{A})$ is the Verdier quotient of the homotopy category $K^b(\mathcal{A})$ by the thick subcategory $K_{ac}^b(\mathcal{A})$ consisting of bounded \mathcal{C} -acyclic complexes. It is in fact a kind of the derived category of an exact category in the sense of A. Neeman [N], and has been studied in different setups, see for examples A. B. Buan [Bu], [GZ], X.W.Chen [C2], and J.Asadollahi, R.Hafezi, and R.Vahed [AHV]. If \mathcal{C} is contravariantly finite in \mathcal{A} , then $D_{\mathcal{C}}^b(\mathcal{A}) \cong K^{-, \mathcal{C}^b}(\mathcal{C})$. See Proposition 5.6 for the details. We observe that the derived category $D^b(\mathcal{A})$ can be described via the relative derived category $D_{\mathcal{C}}^b(\mathcal{A})$ (see Theorem 6.2 below).

Theorem 1 *Let \mathcal{A} be an abelian category with enough projective objects, and \mathcal{C} a resolving contravariantly finite subcategory of \mathcal{A} . Then we have a triangle-equivalence $D^b(\mathcal{A}) \cong D_{\mathcal{C}}^b(\mathcal{A})/K_{ac}^b(\mathcal{C})$.*

Thus, we get a functor $\pi_* : D_{\mathcal{C}}^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$, given by the Verdier functor. As we mentioned before, one can roughly think $D_{\text{perf}}^b(\mathcal{A})$ as $K^b(\mathcal{P})$. Since $D_{\mathcal{C}}^b(\mathcal{A}) \cong K^{-, \mathcal{C}^b}(\mathcal{C})$ and $K^b(\mathcal{P}) \subseteq K^{-, \mathcal{C}^b}(\mathcal{C})$, we have another functor $\pi^* : K^b(\mathcal{P}) \rightarrow D_{\mathcal{C}}^b(\mathcal{A})$, given by the embedding. So, if $D_{\mathcal{C}}^b(\mathcal{A})$ is smooth, plus some other required properties of π_* and π^* , then the triple $(D_{\mathcal{C}}^b(\mathcal{A}), \pi_*, \pi^*)$ can be served as a categorical resolution of $D^b(\mathcal{A})$ in our consideration. We will see below that for some kinds of \mathcal{A} and \mathcal{C} , this machinery works.

1.4. Let A be an Artin algebra, $A\text{-mod}$ (resp. $A\text{-Mod}$) the category of left A -modules (resp. finitely generated left A -modules). We say that A is *representation-finite*, if $A\text{-mod}$ has only finite many pairwise non-isomorphic indecomposable objects. By a theorem of M. Auslander, in this case the indecomposable objects in $A\text{-Mod}$ coincide with the ones indecomposable objects in $A\text{-mod}$ ([A2]). For $T \in A\text{-mod}$, let ${}^{\perp}T$ denote the full subcategory of $A\text{-mod}$ consisting of A -modules X such that $\text{Ext}_A^i(X, T) = 0$, $\forall i \geq 1$. By $\text{add}T$ we denote the full subcategory of $A\text{-mod}$ consisting of the direct summands of finite direct sums of copies of T , and by $\text{Add}T$ the full subcategory of $A\text{-Mod}$ consisting of the direct summands of arbitrary direct sums of copies of T . We say that ${}^{\perp}T$ is finite, if there are only finitely many pairwise non-isomorphic indecomposable A -modules in ${}^{\perp}T$,

or equivalently, there is a module M in $A\text{-mod}$ such that ${}^{\perp}T = \text{add}M$. The following result on the endomorphism algebras of finite global dimension is another key step for the categorical resolution in this paper.

Theorem 2 Let A be an Artin algebra, T and M modules in $A\text{-mod}$ such that ${}^{\perp}T = \text{add}M$. Put $B := (\text{End}_A M)^{op}$. Then for each positive integer $r \geq 2$, $\text{gl.dim} B \leq r$ if and only if $\text{inj.dim} T \leq r$.

Two special cases of Theorem 1 are well-known. If A is representation-finite and T is an injective A -module, then the corresponding B is the Auslander algebra ([ARS]). If A is CM-finite Gorenstein algebra and $T = {}_A A$, then the corresponding B is the relative Auslander algebra of A (see [LZ], [Bel2], and [Leu]).

1.5. Assume that A is of infinite global dimension. If there exist modules T and M in $A\text{-mod}$ with $\text{inj.dim} T < \infty$, such that ${}^{\perp}T = \text{add}M$, then $D^b(B\text{-mod})$ is smooth by Theorem 2, where $B = (\text{End}_A M)^{op}$. While $D^b(B\text{-mod})$ is triangle-equivalent to the relative derived category $D_{\text{add}M}^b(A\text{-mod})$, so by the comment after Theorem 1 we have a triple $(D^b(B\text{-mod}), \pi_*, \pi^*)$. The main results (Theorems 7.1, 7.2 and 7.5) of this paper say that this triple gives a categorical resolution of $D^b(A\text{-mod})$.

Theorem 3. *Let A be an Artin algebra with $\text{gl.dim} A = \infty$.*

(i) *Assume that there are modules T and M in $A\text{-mod}$ with $\text{inj.dim} T < \infty$, such that ${}^{\perp}T = \text{add}M$. Then $D^b(B\text{-mod})$ is a categorical resolution $D^b(A\text{-mod})$, where $B = (\text{End}_A M)^{op}$.*

(ii) *Assume that there are modules T and M in $A\text{-mod}$ with $\text{inj.dim} T < \infty$, such that ${}^{\perp}_{\text{big}}(\text{Add}T) = \text{Add}M$. Then $D^b(B\text{-Mod})$ is a categorical resolution $D^b(A\text{-Mod})$, where $B = (\text{End}_A M)^{op}$.*

(iii) *If A is a CM-finite Gorenstein algebra, and B the relative Auslander algebra of A , then $D^b(B\text{-mod})$ is a weakly crepant categorical resolution of $D^b(A\text{-mod})$, and $D^b(B\text{-Mod})$ is a weakly crepant categorical resolution of $D^b(A\text{-Mod})$.*

We remark that in Theorem 3(iii) if A is in addition a commutative local ring, then G. J. Leuschke [Leu, Section 3] has observed a connection with non-commutative crepant resolution in the sense of M. Van den Bergh [Van].

1.6. The paper is organized as follows. In §2 we recall the main definitions and facts used. For an abelian category \mathcal{A} with enough projective objects, in §3 we give two frequently used cases of \mathcal{A} , such that $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$. In §4 we prove Theorem 2. In §5 we recall some points of the relative derived categories; and in §6 we prove Theorem 1 and other facts, which provide the adjointness of the functors appeared in the categorical resolution. In §7 we prove the main results with some consequences.

2. Preliminaries

2.1. The key formula. Let \mathcal{A} be an abelian category. For $*$ \in {blank, $-$, b }, let $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ be the corresponding homotopy category and the derived category of \mathcal{A} , respectively. For complexes X and Y , let $\text{Hom}_{\mathcal{A}}(X, Y)$ be the Hom complex. Then we have the key formula $\text{Hom}_{K(\mathcal{A})}(X, Y[n]) = H^n \text{Hom}_{\mathcal{A}}(X, Y)$, $\forall n \in \mathbb{Z}$.

2.2. Verdier quotients. Let \mathcal{B} be a triangulated subcategory of triangulated category \mathcal{K} . Thus, in particular, \mathcal{B} is a full subcategory of \mathcal{K} , and closed under isomorphisms $([N])$. By definition a morphism $f : X \rightarrow Y$ of the Verdier quotient \mathcal{K}/\mathcal{B} is an equivalence class of right fractions a/s , where $s : Z \rightarrow X$ and $a : Z \rightarrow Y$ are morphisms of \mathcal{K} , such that “the mapping cone” of s belongs to \mathcal{B} . Let $Q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{B}$ be the localization functor sending an object X to X itself, and sending a morphism $a : X \rightarrow Y$ to a/Id_X . Then Q is a triangle functor with $Q(\mathcal{B}) = 0$; and if $F : \mathcal{K} \rightarrow \mathcal{T}$ is a triangle functor with $G(\mathcal{B}) = 0$, then there is a unique triangle functor $G : \mathcal{K}/\mathcal{B} \rightarrow \mathcal{T}$ such that $F = GQ$. Thus $Q(X) \cong 0$ if and only if X is a direct summand of an object in \mathcal{B} . If \mathcal{B} is *thick* in \mathcal{K} (i.e., \mathcal{B} is a triangulated subcategory of \mathcal{K} which is closed under direct summands), then $Q(X) \cong 0$ if and only if $X \in \mathcal{B}$.

We also need the left fraction construction of \mathcal{K}/\mathcal{B} : a morphism $f : X \rightarrow Y$ in \mathcal{K}/\mathcal{B} is an equivalence class of left fractions $s \backslash a$, where $a : X \rightarrow Z$ and $s : Y \rightarrow Z$ are morphisms of \mathcal{K} , such that “the mapping cone” of s belongs to \mathcal{B} . The localization functor sends a morphism $a : X \rightarrow Y$ to $\text{Id}_Y \backslash a$. Then the Verdier quotient \mathcal{K}/\mathcal{B} constructed via right fractions is isomorphic to the one constructed via left fractions.

Lemma 2.1. ([Ver, Corollary 4-3]) *Let \mathcal{D}_1 and \mathcal{D}_2 be triangulated subcategories of triangulated category \mathcal{C} , and \mathcal{D}_1 a subcategory of \mathcal{D}_2 . Then $\mathcal{D}_2/\mathcal{D}_1$ is a triangulated subcategory of $\mathcal{C}/\mathcal{D}_1$, and $(\mathcal{C}/\mathcal{D}_1)/(\mathcal{D}_2/\mathcal{D}_1) \cong \mathcal{C}/\mathcal{D}_2$ as triangulated categories.*

2.3. Perfect objects of a triangulated category. Let \mathcal{D} be a triangulated category, with shift functor denoted by $[1]$. Following [Kuz], an object $P \in \mathcal{D}$ is *perfect*, provided that for each object Y of \mathcal{D} , there are only finitely many $i \in \mathbb{Z}$ such that $\text{Hom}_{\mathcal{D}}(P, Y[i]) \neq 0$. Denote by $\mathcal{D}_{\text{perf}}$ the full subcategory of \mathcal{D} consisting of perfect objects, which is called *the perfect subcategory* of \mathcal{D} . Then $\mathcal{D}_{\text{perf}}$ is a thick subcategory of \mathcal{D} .

This definition comes from the intrinsic characterization of a perfect complex of $D^b(X)$ of coherent sheaves on algebraic variety X ([O1, Proposition 1.11]). Its advantage is that a triangle-equivalence $\mathcal{D} \rightarrow \mathcal{D}'$ restricts to a triangle-equivalence $\mathcal{D}_{\text{perf}} \rightarrow \mathcal{D}'_{\text{perf}}$, and hence induces a triangle-equivalence $\mathcal{D}/\mathcal{D}_{\text{perf}} \rightarrow \mathcal{D}'/\mathcal{D}'_{\text{perf}}$. For an abelian category \mathcal{A} with enough projective objects, one has $D_{\text{perf}}^b(\mathcal{A}) := D^b(\mathcal{A})_{\text{perf}} \supseteq K^b(\mathcal{P})$, and $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$ in many important cases, where \mathcal{P} is the full subcategory of \mathcal{A} consisting of projective objects. For examples, this is the case when $\mathcal{A} = R\text{-Mod}$ for ring R , or $\mathcal{A} = A\text{-mod}$ for an Artin algebra A . For the details see §3. However, in general we do not know whether $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$.

2.4. Smooth triangulated categories. Let X be an algebraic variety. D. Orlov called the Verdier quotient $D_{sg}^b(X) := D^b(X)/D_{\text{perf}}^b(X)$ the singularity category of X , where $D_{\text{perf}}^b(X) := (D^b(X))_{\text{perf}}$. Then X is smooth if and only if $D_{sg}^b(X) = 0$.

For any abelian category \mathcal{A} , this kind of Verdier quotient was introduced by R. O. Buchweitz [Buch, 1.2.2], under the name *the stabilized derived category* of \mathcal{A} . For short, following [O2], in this paper we call $D_{sg}^b(\mathcal{A}) := D^b(\mathcal{A})/D_{\text{perf}}^b(\mathcal{A})$ *the singularity category* of \mathcal{A} . If \mathcal{A} has enough projective objects and $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$, then $D_{sg}^b(\mathcal{A}) = 0$ if and only if each object of \mathcal{A} has a finite projective dimension. Thus, if $\mathcal{A} = A\text{-mod}$ for an Artin algebra A , then $D_{sg}^b(\mathcal{A}) = 0$ if and only if the global dimension of A is finite.

Definition 2.2. *A triangulated category \mathcal{D} is smooth, if it is triangle-equivalent to $D^b(\mathcal{A})$ with $D_{sg}^b(\mathcal{A}) = 0$, where \mathcal{A} is an abelian category.*

Thus, by the remark at the previous subsection, if a triangulated category \mathcal{D} is smooth, then $\mathcal{D} = \mathcal{D}_{\text{perf}}$. In particular, $D^b(R\text{-Mod})$ is smooth if and only if each R -module has a finite projective dimension.

Definition 2.2 is almost same with a smooth triangulated category in the sense of A. Bondal and D. Orlov [BO, Section 5], where it is triangle-equivalent to $D^b(\mathcal{A})$ with \mathcal{A} an abelian category of finite homological dimension (i.e., for each object $X \in \mathcal{A}$, there are only finite many integers i such that $\text{Ext}_{\mathcal{A}}^i(X, -) \neq 0$). In fact, if \mathcal{A} has enough projective objects, then a smooth triangulated category in the sense of [BO] is smooth in the sense of Definition 2.2; and the converse is also true if $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$. The reason to make such a minor change is that we do not know in general the invariance of finite homological dimension under derived equivalences, although this is true in many important cases.

Definition 2.2 is also a slight modification of [Kuz, Definition 3.1], where a smooth triangulated category is triangle-equivalent to an admissible subcategory of $D^b(X)$ with $D_{sg}^b(X) = 0$, where X is an algebraic variety. For our purpose we do not need to consider admissible subcategories.

2.5. Categorical resolution of a triangulated category. For a triangle functor $F : \mathcal{T} \rightarrow \mathcal{T}'$, let $\text{Ker} F$ denote the full subcategory of \mathcal{T} consisting of objects K with $F(K) \cong 0$. The following definition is due to A. Bondal and D. Orlov [BO, Section 5] and A. Kuznetsov [Kuz, Definition 3.2].

Definition 2.3. *A categorical resolution of a non-smooth triangulated category \mathcal{D} is a smooth triangulated category $\tilde{\mathcal{D}}$, or more precisely, a triple $(\tilde{\mathcal{D}}, \pi_*, \pi^*)$, where $\pi_* : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ and $\pi^* : \mathcal{D}_{\text{perf}} \rightarrow \tilde{\mathcal{D}}$ are triangle functors, such that*

- (i) π_* induces a triangle-equivalence $\tilde{\mathcal{D}}/\text{Ker} \pi_* \cong \mathcal{D}$;
- (ii) π^* is left adjoint to π_* on $\mathcal{D}_{\text{perf}}$, that is, there is a functorial isomorphism $\eta_{P,X} : \text{Hom}_{\tilde{\mathcal{D}}}(\pi^* P, X) \cong \text{Hom}_{\mathcal{D}}(P, \pi_* X)$, $\forall P \in \mathcal{D}_{\text{perf}}, \forall X \in \tilde{\mathcal{D}}$;
- (iii) The unit $\eta = (\eta_P)_{P \in \mathcal{D}_{\text{perf}}} : \text{Id}_{\mathcal{D}_{\text{perf}}} \rightarrow \pi_* \pi^*$ is a natural isomorphism of functors, where η_P is the morphism $\eta_{P, \pi^* P}(\text{Id}_{\pi^* P}) : P \rightarrow \pi_* \pi^* P$ in \mathcal{D} .

Note that (ii) implies that $\pi^* : \mathcal{D}_{\text{perf}} \rightarrow \tilde{\mathcal{D}}$ is fully faithful.

If $\pi_* : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ is full and dense, then (i) in Definition 2.3 holds automatically. However π_* usually can not be full.

It is well-known that for a complex singular variety X there is a proper birational resolution of singularities $\tilde{X} \rightarrow X$; and that if $\tilde{X} \rightarrow X$ is a proper birational resolution

of algebraic variety X of rational singularity, then $D^b(\tilde{X})$ is a categorical resolution of $D^b(X)$ in the sense of Definition 2.3.

Definition 2.4. ([Kuz], Definition 3.4) *A categorical resolution $(\tilde{\mathcal{D}}, \pi_*, \pi^*)$ of a triangulated category \mathcal{D} is weakly crepant if π^* is right adjoint to π_* on $\mathcal{D}_{\text{perf}}$, that is, there is a functorial isomorphism $\text{Hom}_{\tilde{\mathcal{D}}}(X, \pi^*P) \cong \text{Hom}_{\mathcal{D}}(\pi_*X, P)$, $\forall P \in \mathcal{D}_{\text{perf}}, \forall X \in \tilde{\mathcal{D}}$.*

A non-commutative crepant resolution ([Van]) induces a weakly crepant categorical resolution of a triangulated category.

2.6. Gorenstein-projective objects. Let \mathcal{A} be an abelian category with enough projective objects, and $\mathcal{P} = \mathcal{P}(\mathcal{A})$ the full subcategory of \mathcal{A} consisting of projective objects. A complete \mathcal{A} -projective resolution is an exact sequence $P^\bullet = \cdots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots$ with each $P^i \in \mathcal{P}$, such that $\text{Hom}_{\mathcal{A}}(P^\bullet, P)$ stays exact for each $P \in \mathcal{P}$. An object G of \mathcal{A} is *Gorenstein-projective* if there is a complete \mathcal{A} -projective resolution P^\bullet such that $G \cong \text{Im} d^0$ (E. E. Enochs and O. M. G. Jenda [EJ]). Denote by $\mathcal{GP}(\mathcal{A})$ the full subcategory of \mathcal{A} consisting of Gorenstein-projective objects.

A full subcategory \mathcal{X} of \mathcal{A} is *resolving* ([AB]), provided that $\mathcal{X} \supseteq \mathcal{P}$, \mathcal{X} is closed under extensions and direct summands, and that \mathcal{X} is closed under the kernels of epimorphisms. A resolving subcategory is of course additive. Then $\mathcal{GP}(\mathcal{A})$ is resolving; and $\mathcal{GP}(\mathcal{A})$ is closed under arbitrary direct sums if \mathcal{A} is *cocomplete*, i.e., \mathcal{A} has arbitrary direct sums (see [AR] and [Hol]).

A *Frobenius category* \mathcal{B} is an exact category ([Q, §2]) with enough projective objects and enough injective objects, such that an object is projective if and only if it is injective (see [K1]). An important feature is that $\mathcal{GP}(\mathcal{A})$ is a Frobenius category, where the projective-injective objects of $\mathcal{GP}(\mathcal{A})$ are exactly the projective objects of \mathcal{A} (see [Bel1]). Thus the stable category $\underline{\mathcal{GP}(\mathcal{A})}$ of $\mathcal{GP}(\mathcal{A})$ modulo \mathcal{P} is triangulated ([Hap, p.16]).

Recall that an Artin algebra A is *CM-finite*, if $\mathcal{GP}(A\text{-mod})$ has only finitely many pairwise non-isomorphic indecomposable objects; and that A is *Gorenstein*, if $\text{inj.dim}_A A < \infty$ and $\text{inj.dim}_A A_A < \infty$. If this is the case, then $\text{inj.dim}_A A = \text{inj.dim}_A A_A$ ([I]). For a Gorenstein algebra A , we have $\mathcal{GP}(A\text{-mod}) = {}^\perp({}_A A)$ (see [EJ, Corollary 11.5.3]; or [Z, Lemma 2.4(iii)] for a short argument). If A is a CM-finite Gorenstein algebra, then the indecomposable objects of $\mathcal{GP}(A\text{-Mod})$ coincide with the indecomposable objects of $\mathcal{GP}(A\text{-mod})$, by X. W. Chen ([C1]).

2.7. Contravariantly finite subcategories. Let \mathcal{B} be an additive category, \mathcal{C} a full additive subcategory of \mathcal{B} , and $X \in \mathcal{B}$. A morphism $f : C \rightarrow X$ with $C \in \mathcal{C}$ is a *right \mathcal{C} -approximation* of X , if $\text{Hom}_{\mathcal{B}}(C', f) : \text{Hom}_{\mathcal{B}}(C', C) \rightarrow \text{Hom}_{\mathcal{B}}(C', X)$ is surjective for each $C' \in \mathcal{C}$. If each object $X \in \mathcal{B}$ admits a right \mathcal{C} -approximation, then \mathcal{C} is said to be *contravariantly finite in \mathcal{B}* ([AR]).

Example 2.5. Recall the contravariantly finite subcategories used in this paper.

(i) Let A be an Artin algebra and $M \in A\text{-mod}$. Then $\text{add}M$ is contravariantly finite in $A\text{-mod}$; and $\text{Add}M$ is contravariantly finite in $A\text{-Mod}$ (here M is also assumed to be finitely generated).

(ii) If each object of \mathcal{A} has a finite Gorenstein-projective dimension, then $\mathcal{GP}(\mathcal{A})$ is contravariantly finite in \mathcal{A} ([EJ, Theorem 11.5.1], or [Hol, Theorem 2.10]).

(iii) For an Artin algebra A , $\mathcal{GP}(A\text{-Mod})$ is contravariantly finite in $A\text{-Mod}$ ([Bel1, Theorem 3.5]).

(iv) An Artin algebra A is Gorenstein if and only if each A -module has a finite Gorenstein-projective dimension in $A\text{-Mod}$, also if and only if each finitely generated A -module has a finite Gorenstein-projective dimension in $A\text{-mod}$ ([Hos]). Thus, if A is Gorenstein, then $\mathcal{GP}(A\text{-mod})$ is contravariantly finite in $A\text{-mod}$.

(v) A. Beligiannis [Bel1] introduced virtually Gorenstein algebras. A Gorenstein algebra is virtually Gorenstein, but the converse is not true. However, for a virtually Gorenstein algebra A , $\mathcal{GP}(A\text{-mod})$ is contravariantly finite in $A\text{-mod}$ ([Bel1, Theorem 8.2(ix)]).

(vi) For examples of CM-finite non-Gorenstein algebras we refer to [Rin]. For a CM-finite algebra A , $\mathcal{GP}(A\text{-mod})$ is contravariantly finite in $A\text{-mod}$.

3. Perfect subcategory of a triangulated category

Throughout this section, \mathcal{A} is an abelian category with enough projective objects, $\mathcal{P} = \mathcal{P}(\mathcal{A})$ the full subcategory of \mathcal{A} consisting of projective objects. We give two classes of \mathcal{A} , such that $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$.

3.1. The following characterization of objects in $K^b(\mathcal{P})$ is due to Buchweitz. It also implies that $K^b(\mathcal{P})$ is thick in $D^b(\mathcal{A})$.

Lemma 3.1. ([Buch, Lemma 1.2.1]) *Let \mathcal{A} be an abelian category with enough projective objects, and $P \in D^b(\mathcal{A})$. Then the following are equivalent*

- (i) $P \in K^b(\mathcal{P})$;
- (ii) there is an integer $i(P)$ such that $\text{Hom}_{D^b(\mathcal{A})}(P, M[i]) = 0$ for each $i \geq i(P)$ and for each object M of \mathcal{A} ;
- (iii) there is a finite subset $I(P) \subseteq \mathbb{Z}$, such that $\text{Hom}_{D^b(\mathcal{A})}(P, M[j]) = 0$ for each $j \notin I(P)$ and for each object M of \mathcal{A} .

Proof. For convenience of the reader, we include an argument for $(iii) \implies (i)$. Let $Q \longrightarrow P$ be a quasi-isomorphism with $Q \in K^{-,b}(\mathcal{P})$. Then there is an integer N such that $H^n Q = 0$ for all $n \leq N$. We claim that there exists an integer n with $n \leq N$ such that $\text{Im} d_Q^n \in \mathcal{P}$. If this claim is true, then there is a quasi-isomorphism

$$\begin{array}{ccccccc}
 Q : & \cdots & \longrightarrow & Q^{n-1} & \longrightarrow & Q^n & \longrightarrow & Q^{n+1} & \longrightarrow & \cdots \\
 \downarrow f & & & \downarrow & & \downarrow & & \parallel & & \\
 \tau_{\geq n+1} Q : & \cdots & \longrightarrow & 0 & \longrightarrow & \text{Im } d_Q^n & \longrightarrow & Q^{n+1} & \longrightarrow & \cdots
 \end{array}$$

Therefore we have isomorphisms $P \cong Q \cong \tau_{\geq n+1} Q \in K^b(\mathcal{P})$ in $D^-(\mathcal{A})$, and hence we have an isomorphism $P \cong \tau_{\geq n+1} Q \in K^b(\mathcal{P})$ in $D^b(\mathcal{A})$.

Assume that the claim is not true. Then there exists $-n \notin I(P)$ such that $M := \text{Im}d_Q^n \notin \mathcal{P}$. Denote by $\widetilde{d}^n : Q^n \rightarrow M = \text{Im}d_Q^n$ the epimorphism induced by d_Q^n . Then

$$\widetilde{d}^n \in \text{Ker}(\text{Hom}_{\mathcal{A}}(Q^n, M) \xrightarrow{\text{Hom}_{\mathcal{A}}(d^{n-1}, M)} \text{Hom}_{\mathcal{A}}(Q^{n-1}, M));$$

but

$$\widetilde{d}^n \notin \text{Im}(\text{Hom}_{\mathcal{A}}(Q^{n+1}, M) \xrightarrow{\text{Hom}_{\mathcal{A}}(d^n, M)} \text{Hom}_{\mathcal{A}}(Q^n, M))$$

(otherwise $\text{Im}d^n \hookrightarrow Q^{n+1}$ splits, which contradicts the assumption $M \notin \mathcal{P}$). This implies $H^{-n}\text{Hom}_{\mathcal{A}}(Q, M) \neq 0$. Therefore

$$\begin{aligned} \text{Hom}_{D^b(\mathcal{A})}(P, M[-n]) &\cong \text{Hom}_{D^-(\mathcal{A})}(Q, M[-n]) \\ &\cong \text{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \\ &= H^{-n}\text{Hom}_{\mathcal{A}}(Q, M) \neq 0. \end{aligned}$$

This contradicts the assumption (iii). ■

3.2. We say that \mathcal{A} is a *finitely filtrated category*, if there exists finitely many objects S_1, \dots, S_m , such that for any non-zero object X of \mathcal{A} , there exists a sequence of monomorphisms

$$0 = X_0 \xrightarrow{f_0} \dots \rightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n = X$$

such that $\text{Coker}f_i \in \{S_1, \dots, S_m\}$, $0 \leq i \leq n-1$.

For example, for an Artin algebra A , $A\text{-mod}$ is finitely filtrated.

Proposition 3.2. *Let \mathcal{A} be a finitely filtrated category. Then $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$.*

Proof. We only justify $D_{\text{perf}}^b(\mathcal{A}) \subseteq K^b(\mathcal{P})$. Let $P \in D_{\text{perf}}^b(\mathcal{A})$. By assumption \mathcal{A} is finitely filtrated by some objects S_1, \dots, S_m . Put $S := S_1 \oplus \dots \oplus S_m$. Since $P \in D_{\text{perf}}^b(\mathcal{A})$, there are only finitely many $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(\mathcal{A})}(P, S[i]) \neq 0$. Denote by $I(P)$ the finite set of such integers i 's. Then $\text{Hom}_{D^b(\mathcal{A})}(P, S[j]) = 0$ for $j \notin I(P)$. Since each object $M \in \mathcal{A}$ has a filtration with factors belonging to $\{S_1, \dots, S_m\}$, and since each short exact sequence in \mathcal{A} gives rise a distinguished triangle in $D^b(\mathcal{A})$, it follows that $\text{Hom}_{D^b(\mathcal{A})}(P, M[j]) = 0$ for each $j \notin I(P)$ and for each object M of \mathcal{A} . Thus $P \in K^b(\mathcal{P})$ by Lemma 3.1. ■

3.3. If \mathcal{A} has arbitrary direct sums, then we have the same conclusion as in Proposition 3.2. It in particular say $D_{\text{perf}}^b(R\text{-Mod}) = K^b(\mathcal{P}(R\text{-Mod}))$, where R is a ring.

Proposition 3.3. *Let \mathcal{A} be a cocomplete abelian category with enough projective objects. Then $D_{\text{perf}}^b(\mathcal{A}) = K^b(\mathcal{P})$.*

Proof. We only prove $D_{\text{perf}}^b(\mathcal{A}) \subseteq K^b(\mathcal{P})$. The idea of the proof could be found from J. Rickard [Ric, Proposition 6.2]. Let $P \in D_{\text{perf}}^b(\mathcal{A})$. Take a quasi-isomorphism $Q \rightarrow P$ with $Q \in K^{-b}(\mathcal{P})$. Thus, there is an $N \in \mathbb{Z}$ such that $H^n P = 0$, $\forall n \leq N$. As in the proof of Proposition 3.1 it suffices to prove that there exists an integer n with $n \leq N$ such that $\text{Im}d_Q^n \in \mathcal{P}$.

Otherwise, $\text{Im}d_Q^n \notin \mathcal{P}$ for each $n \leq N$. Since \mathcal{A} has infinite direct sums, we could put $M := \bigoplus_{n \leq N} \text{Im}d_Q^n \in \mathcal{A}$. Since $\text{Im}d_Q^n \neq 0$, we have a non-zero epimorphism $\widetilde{d}^n : Q^n \rightarrow \text{Im}d_Q^n$, which induces a non-zero morphism

$$f : Q^n \rightarrow M = \bigoplus_{j \leq N} \text{Im}d_Q^j = \text{Im}d_Q^n \oplus \left(\bigoplus_{j \leq N, j \neq n} \text{Im}d_Q^j \right).$$

Clearly f induces a chain map $Q \rightarrow M[-n]$. Since $\text{Im}d_Q^n \notin \mathcal{P}$, it follows that this chain map is not null homotopic. This shows $\text{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0$ for each integer n with $n \leq N$, and hence

$$\begin{aligned} \text{Hom}_{D^b(\mathcal{A})}(P, M[-n]) &\cong \text{Hom}_{D^-(\mathcal{A})}(Q, M[-n]) \\ &\cong \text{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0. \end{aligned}$$

In other words, we get infinitely many integers i such that $\text{Hom}_{D^b(\mathcal{A})}(P, M[i]) \neq 0$. This contradicts the assumption $P \in D_{\text{perf}}^b(\mathcal{A})$. \blacksquare

4. Global dimension of a class of endomorphism algebras

4.1. Let \mathcal{A} be an abelian category with enough projective objects, and X an object of \mathcal{A} . The global dimension $\text{gl.dim} \mathcal{A}$ is the supreme of the projective dimension $\text{proj.dim} X$, where X runs over all the objects of \mathcal{A} . For a ring R , $\text{gl.dim}(R\text{-Mod})$ is exactly the supreme of $\text{proj.dim} M$, where M runs over all the cyclic left R -modules (see [A, Theorem 1]). Thus, if R is left noetherian, then $\text{gl.dim}(R\text{-Mod}) = \text{gl.dim}(R\text{-mod})$, which will be denoted by $\text{gl.dim} R$. Thus, for Artin algebra A , $\text{gl.dim} A$ is just the maximum of $\text{proj.dim} S(i)$, $1 \leq i \leq n$, where $\{S(1), \dots, S(n)\}$ is a complete set of pairwise non-isomorphic simple A -modules.

4.2. Let A be an Artin algebra, and $M \in A\text{-mod}$. The functor $\text{Hom}_A(M, -) : A\text{-mod} \rightarrow B\text{-mod}$ induces an equivalence between $\text{add } M$ and $\mathcal{P}(B\text{-mod})$, where $B = (\text{End}_A M)^{\text{op}}$ ([ARS, p.33]). If M is a generator (i.e., ${}_A A \in \text{add } M$), then we have

Lemma 4.1. *Let M be a generator of $A\text{-mod}$. Then $\text{Hom}_A(M, -) : A\text{-mod} \rightarrow B\text{-mod}$ is fully faithful.*

Proof. Since M is a generator, for any $X \in A\text{-mod}$ there is a surjective A -map $M^m \rightarrow X$ for some positive integer m . This implies that $\text{Hom}_A(M, -)$ is faithful.

Let $X, Y \in A\text{-mod}$, and $f : \text{Hom}_A(M, X) \rightarrow \text{Hom}_A(M, Y)$ be a B -map. By taking right $\text{add } M$ -approximations, we get exact sequences $T_1 \xrightarrow{u} T_0 \xrightarrow{\pi} X \rightarrow 0$ and $T'_1 \xrightarrow{u'} T'_0 \xrightarrow{\pi'} Y \rightarrow 0$ with $T_0, T_1, T'_0, T'_1 \in \text{add } M$ (since M is a generator, π and π' are surjective). Applying $\text{Hom}_A(M, -)$ we have the following diagram with exact rows

$$\begin{array}{ccccccc} \text{Hom}_A(M, T_1) & \xrightarrow{\text{Hom}_A(M, u)} & \text{Hom}_A(M, T_0) & \xrightarrow{\text{Hom}_A(M, \pi)} & \text{Hom}_A(M, X) & \longrightarrow & 0 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ \text{Hom}_A(M, T'_1) & \xrightarrow{\text{Hom}_A(M, u')} & \text{Hom}_A(M, T'_0) & \xrightarrow{\text{Hom}_A(M, \pi')} & \text{Hom}_A(M, Y) & \longrightarrow & 0. \end{array}$$

Then f induces f_1 and f_0 such that the above diagram commutes. Thus $f_i = \text{Hom}_A(M, f'_i)$ for some $f'_i \in \text{Hom}_A(T_i, T'_i)$, $i = 0, 1$. So we get the following diagram

$$\begin{array}{ccccccc} T_1 & \xrightarrow{u} & T_0 & \xrightarrow{\pi} & X & \longrightarrow & 0 \\ \downarrow f'_1 & & \downarrow f'_0 & & \downarrow f' & & \\ T'_1 & \xrightarrow{u'} & T'_0 & \xrightarrow{\pi'} & Y & \longrightarrow & 0 \end{array}$$

with commutative left square. So there exists $f' \in \text{Hom}_A(X, Y)$ such that the above diagram commutes. Thus $f \text{Hom}_A(M, \pi) = \text{Hom}_A(M, f') \text{Hom}_A(M, \pi)$. Since $\text{Hom}_A(M, \pi)$ is surjective, it follows that $f = \text{Hom}_A(M, f')$, i.e., $\text{Hom}_A(M, -)$ is full. \blacksquare

4.3. The following Auslander-Bridger Lemma is very useful.

Lemma 4.2. ([AB, Lemma 3.12]) *Let \mathcal{A} be an abelian category with enough projective objects, \mathcal{X} a resolving subcategory of \mathcal{A} . Assume that*

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow A \longrightarrow 0$$

are exact sequences in \mathcal{A} , such that $X_i \in \mathcal{X}$ and $Y_i \in \mathcal{X}$ for $0 \leq i \leq n-1$. Then $X_n \in \mathcal{X}$ if and only if $Y_n \in \mathcal{X}$.

Proof. We include a shorter proof. Assume $X_n \in \mathcal{X}$. Take an exact sequence in \mathcal{A}

$$0 \longrightarrow K \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0$$

such that P_i are projective, $i = 0, \dots, n-1$. Then we get chain map f^\bullet :

$$\begin{array}{ccccccccccc} P^\bullet : & 0 & \longrightarrow & K & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \longrightarrow & P_0 & \xrightarrow{d_0} & A & \longrightarrow & 0 \\ f^\bullet \downarrow & & & f_n \downarrow & & f_{n-1} \downarrow & & & & f_0 \downarrow & & \parallel & & \\ X^\bullet : & 0 & \longrightarrow & X_n & \xrightarrow{\partial_n} & X_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \longrightarrow & X_0 & \xrightarrow{\partial_0} & A & \longrightarrow & 0. \end{array}$$

Consider the short exact sequence of complexes over \mathcal{A}

$$0 \longrightarrow X^\bullet \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \text{Con}(f^\bullet) \xrightarrow{(1,0)} P^\bullet[1] \longrightarrow 0,$$

where $\text{Con}(f^\bullet)$ denotes the mapping cone of f^\bullet , which is by definition the complex

$$\text{Con}(f^\bullet) : \quad 0 \longrightarrow K \xrightarrow{\begin{pmatrix} -d_n \\ f_n \end{pmatrix}} P_{n-1} \oplus X_n \xrightarrow{\begin{pmatrix} -d_{n-1} & 0 \\ f_{n-1} & \partial_n \end{pmatrix}} P_{n-2} \oplus X_{n-1}$$

$$\longrightarrow \cdots \longrightarrow P_0 \oplus X_1 \xrightarrow{\begin{pmatrix} -d_0 & 0 \\ f_0 & \partial_1 \end{pmatrix}} A \oplus X_0 \xrightarrow{(\text{Id}_A, \partial_0)} A \longrightarrow 0.$$

Then $\text{Con}(f^\bullet)$ is again acyclic. Since $(\text{Id}_A, \partial_0)$ is a splitting epimorphism, it follows that we get an acyclic complex

$$0 \longrightarrow K \longrightarrow P_{n-1} \oplus X_n \longrightarrow P_{n-2} \oplus X_{n-1} \longrightarrow \cdots \longrightarrow P_0 \oplus X_1 \xrightarrow{(f_0, \partial_1)} X_0 \longrightarrow 0$$

where $X_0 \in \mathcal{X}$ and $P_{i-1} \oplus X_i \in \mathcal{X}$ for $i = 1, \dots, n$. Since \mathcal{X} is closed under taking kernels of epimorphisms, it follows that $K \in \mathcal{X}$.

By the similar way we get an acyclic complex

$$0 \longrightarrow K \longrightarrow P_{n-1} \oplus Y_n \xrightarrow{\alpha} P_{n-2} \oplus Y_{n-1} \longrightarrow \cdots \longrightarrow P_0 \oplus Y_1 \longrightarrow Y_0 \longrightarrow 0$$

where $Y_0 \in \mathcal{X}$, $P_{i-1} \oplus Y_i \in \mathcal{X}$ for $i = 1, \dots, n-1$, and $\text{Im} \alpha \in \mathcal{X}$. Since \mathcal{X} is closed under taking extensions, by the short exact sequence $0 \longrightarrow K \longrightarrow P_{n-1} \oplus Y_n \longrightarrow \text{Im} \alpha \longrightarrow 0$ with $K \in \mathcal{X}$ and $\text{Im} \alpha \in \mathcal{X}$, we know $P_{n-1} \oplus Y_n \in \mathcal{X}$. Since \mathcal{X} is closed under taking direct summands, we have $Y_n \in \mathcal{X}$. \blacksquare

4.4. The following result is one of the key steps in our categorical resolutions.

Theorem 4.3. *Let A be an Artin algebra, T and M modules in $A\text{-mod}$ such that ${}^\perp T = \text{add} M$. Put $B := (\text{End}_A M)^{\text{op}}$. Then for each positive integer $r \geq 2$, $\text{gl.dim} B \leq r$ if and only if $\text{inj.dim} T \leq r$.*

Proof. Assume that $\text{gl.dim} B \leq r$. Let $X \in A\text{-mod}$. Consider a right $\text{add} M$ -approximation $f_0 : M_0 \longrightarrow X$. Since M is a generator, f_0 is surjective. Again considering a right $\text{add} M$ -approximation $M_1 \longrightarrow \text{Ker} f_0$ and continuing this process we get an exact sequence in $A\text{-mod}$

$$M_{r-1} \xrightarrow{f_{r-1}} M_{r-2} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $M_i \in \text{add} M$, $0 \leq i \leq r-1$. Put $K := \text{Ker} f_{r-1}$. By construction we get an exact sequence

$$0 \rightarrow \text{Hom}(M, K) \longrightarrow \text{Hom}(M, M_{r-1}) \longrightarrow \cdots \longrightarrow \text{Hom}(M, M_0) \longrightarrow \text{Hom}(M, X) \rightarrow 0.$$

Since by assumption $\text{proj.dim}_B \text{Hom}(M, X) \leq r$, it follows from Auslander-Bridger Lemma that $\text{Hom}_A(M, K)$ is projective as a B -module. Thus there is a B -isomorphism $s : \text{Hom}_A(M, K) \longrightarrow \text{Hom}_A(M, M')$ with $M' \in \text{add} M$. By Lemma 4.1 there exists $f : K \longrightarrow M'$ and $g : M' \longrightarrow K$ such that $s = \text{Hom}_A(M, f)$ and $s^{-1} = \text{Hom}_A(M, g)$. Therefore

$$\text{Hom}_A(M, fg) = \text{Id}_{\text{Hom}_A(M, M')} = \text{Hom}_A(M, \text{Id}_{M'}).$$

Since $\text{Hom}_A(M, -)$ is faithful, it follows that $fg = \text{Id}_{M'}$. Thus $K \in \text{add} M$, and hence we have an exact sequence in $A\text{-mod}$

$$0 \longrightarrow M_r \longrightarrow M_{r-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

with $M_i \in \text{add} M$, $0 \leq i \leq r$. For $i \geq 1$, by dimension shift we have

$$\text{Ext}_A^{i+r}(X, T) \cong \text{Ext}_A^i(M_r, T) = 0$$

since $M \in {}^\perp T$. This shows $\text{inj.dim} T \leq r$.

Note that the argument above holds also for $r \leq 1$.

Conversely, assume that $\text{inj.dim} T \leq r$ with $r \geq 2$. Let ${}_B Y \in B\text{-mod}$. Taking a projective presentation of ${}_B Y$

$$\text{Hom}_A(M, M_1) \xrightarrow{d} \text{Hom}_A(M, M_0) \longrightarrow {}_B Y \longrightarrow 0$$

with $M_i \in \text{add}M$, $i = 0, 1$. Then there exists an A -map $f : M_1 \rightarrow M_0$ such that $d = \text{Hom}_A(M, f)$. Considering a right $\text{add}M$ -approximation $M_2 \rightarrow \text{Ker}f$ and continuing this process, we get an exact sequence in $A\text{-mod}$

$$0 \rightarrow K \rightarrow M_{r-1} \rightarrow \cdots \rightarrow M_2 \rightarrow M_1 \xrightarrow{f} M_0$$

with $M_i \in \text{add}M$, $0 \leq i \leq n-1$. For $i \geq 1$, by dimension shift we have

$$\text{Ext}_A^i(K, T) \cong \text{Ext}_A^{i+r-2}(\text{Ker}f, T) \cong \text{Ext}_A^{i+r-1}(\text{Im}f, T) \cong \text{Ext}_A^{i+r}(\text{Coker}f, T) = 0$$

since $\text{inj.dim}T \leq r$. Thus $K \in {}^\perp T = \text{add}M$. By construction we get an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_A(M, K) \rightarrow \text{Hom}_A(M, M_{r-1}) \rightarrow \cdots \rightarrow \text{Hom}_A(M, M_2) \\ \rightarrow \text{Hom}_A(M, M_1) \xrightarrow{d} \text{Hom}_A(M, M_0) \rightarrow {}_B Y \rightarrow 0. \end{aligned}$$

This gives a projective resolution of ${}_B Y$, and hence $\text{proj.dim}_B Y \leq r$. This proves $\text{gl.dim}B \leq r$. \blacksquare

Remark 4.4. (i) If A is representation-finite and T an injective module, then Theorem 4.3 is Auslander's result, and the corresponding B is the Auslander algebra ([ARS]).

(ii) If A is CM-finite Gorenstein algebra and $T = {}_A A$, then Theorem 4.3 is also well-known (see [LZ] and [Bel2]; also [Leu]), and the corresponding B is the relative Auslander algebra of A .

5. Relative derived categories

For our purpose we recall some points of relative derived categories. Let \mathcal{C} be a full additive subcategory of an abelian category \mathcal{A} .

5.1. A complex M^\bullet over \mathcal{A} is \mathcal{C} -acyclic, if $\text{Hom}_{\mathcal{A}}(C, M^\bullet)$ is acyclic for all objects C in \mathcal{C} . A chain map $f^\bullet : X^\bullet \rightarrow Y^\bullet$ is a \mathcal{C} -quasi-isomorphism, if $\text{Hom}_{\mathcal{A}}(C, f^\bullet) : \text{Hom}_{\mathcal{A}}(C, X^\bullet) \rightarrow \text{Hom}_{\mathcal{A}}(C, Y^\bullet)$ is a quasi-isomorphism for all objects C in \mathcal{C} . Then f^\bullet is a \mathcal{C} -quasi-isomorphism if and only if the mapping cone $\text{Con}(f^\bullet)$ is \mathcal{C} -acyclic.

For $*$ in $\{b, -, \text{blank}\}$, let $K_{\mathcal{C}ac}^*(\mathcal{A})$ denote the full subcategory of the homotopy category $K^*(\mathcal{A})$ consisting of \mathcal{C} -acyclic complexes. Then

$$K_{\mathcal{C}ac}^*(\mathcal{A}) = {}^\perp \mathcal{C} := \{X^\bullet \in K^*(\mathcal{A}) \mid \text{Hom}_{K^*(\mathcal{A})}(C, X^\bullet[n]) = 0, \forall n \in \mathbb{Z}, \forall C \in \mathcal{C}\}.$$

Thus $K_{\mathcal{C}ac}^*(\mathcal{A})$ is a thick subcategory of $K^*(\mathcal{A})$. The Verdier quotient

$$D_{\mathcal{C}}^*(\mathcal{A}) := K^*(\mathcal{A})/K_{\mathcal{C}ac}^*(\mathcal{A})$$

is called the \mathcal{C} -relative derived category. See [GZ], [C2] and [AHV].

Example 5.1. (i) If \mathcal{A} has enough projective objects with \mathcal{P} the full subcategory consisting of projective objects, and $\mathcal{C} = \mathcal{P}$, then $D_{\mathcal{C}}^*(\mathcal{A})$ is just the derived category $D^*(\mathcal{A})$.

(ii) Let \mathcal{A} be as in (i). If $\mathcal{C} = \mathcal{GP}(\mathcal{A})$, the full subcategory of the Gorenstein-projective objects of \mathcal{A} , then $D_{\mathcal{C}}^*(\mathcal{A})$ is the Gorenstein derived category in [GZ].

(iii) Let A be an Artin algebra, and $M \in A\text{-mod}$. Then we have the M -relative derived categories $D_{\text{add}M}^*(A\text{-mod})$ and $D_{\text{Add}M}^*(A\text{-Mod})$. See [AHV].

5.2. It is important that the upper bounded derived category $D^-(\mathcal{A})$ is a triangulated subcategory of the unbounded derived category $D(\mathcal{A})$, and that the bounded derived category $D^b(\mathcal{A})$ is a triangulated subcategory of the $D^-(\mathcal{A})$. The \mathcal{C} -relative derived category enjoy this property. The proof is similar as the Gorenstein derived category ([GZ, 2.5]), with a minor change. For the convenience of the reader we include the proof.

Lemma 5.2. ([K2], Lemma 10.3) *Let \mathcal{B} and \mathcal{D} be triangulated subcategories of triangulated category \mathcal{C} . If one of the following conditions is satisfied, then the canonical triangle functor $\mathcal{D}/\mathcal{D} \cap \mathcal{B} \rightarrow \mathcal{C}/\mathcal{B}$ is fully faithful.*

(i) *Each morphism $X \rightarrow B$ with $B \in \mathcal{B}$ and $X \in \mathcal{D}$ admits a factorization $X \rightarrow B' \rightarrow B$ with $B' \in \mathcal{D} \cap \mathcal{B}$.*

(ii) *Each morphism $B \rightarrow Y$ with $B \in \mathcal{B}$ and $Y \in \mathcal{D}$ admits a factorization $B \rightarrow B' \rightarrow Y$ with $B' \in \mathcal{D} \cap \mathcal{B}$.*

Proposition 5.3. *Let \mathcal{C} be a full additive subcategory of an abelian category \mathcal{A} . Then $D_{\mathcal{C}}^-(\mathcal{A})$ is a triangulated subcategory of $D_{\mathcal{C}}(\mathcal{A})$; and $D_{\mathcal{C}}^b(\mathcal{A})$ is a triangulated subcategory of $D_{\mathcal{C}}^-(\mathcal{A})$, and hence of $D_{\mathcal{C}}(\mathcal{A})$.*

Proof. We prove the first assertion by Lemma 5.2(i), the second one can be proved by Lemma 5.2(ii). Let $f^\bullet : X^\bullet \rightarrow B^\bullet$ be a chain map with $B^\bullet \in K_{\mathcal{C}ac}(\mathcal{A})$ and $X^\bullet \in K^-(\mathcal{A})$. We may assume that $X^i = 0$ for $i > 0$. Then f^\bullet admits the following natural factorization:

$$\begin{array}{ccccccc} X^\bullet : & \cdots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ f^\bullet \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ B'^\bullet : & \cdots & \longrightarrow & B^{-1} & \longrightarrow & B^0 & \longrightarrow & \text{Ker } d^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow & & & \parallel & & \parallel & & \downarrow & & \downarrow & & \\ B^\bullet : & \cdots & \longrightarrow & B^{-1} & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & \cdots \end{array}$$

We need to prove that B'^\bullet is \mathcal{C} -acyclic. Since B^\bullet is \mathcal{C} -acyclic, it suffices to prove that

$$\text{Hom}_{\mathcal{A}}(C, B^{-1}) \xrightarrow{d_*^{-1}} \text{Hom}_{\mathcal{A}}(C, B^0) \xrightarrow{\tilde{d}_*^0} \text{Hom}_{\mathcal{A}}(C, \text{Ker } d^1) \rightarrow 0$$

is exact for each $C \in \mathcal{C}$, where $\tilde{d}^0 : B^0 \rightarrow \text{Ker } d^1$ is induced by d^0 . Since $0 \rightarrow \text{Hom}(C, \text{Ker } d^1) \xrightarrow{\sigma} \text{Hom}(C, B^1) \rightarrow \text{Hom}(C, B^2)$ is exact, by the commutative diagram

$$\begin{array}{ccccc} \text{Hom}(C, B^{-1}) & \xrightarrow{d_*^{-1}} & \text{Hom}(C, B^0) & \xrightarrow{d_*^0} & \text{Hom}(C, B^1) & \xrightarrow{d_*^1} & \text{Hom}(C, B^2) \\ & & \searrow \tilde{d}_*^0 & & \nearrow \sigma & & \\ & & \text{Hom}(C, \text{Ker } d^1) & & & & \end{array}$$

we have $\text{Ker } \tilde{d}_*^0 = \text{Ker } d_*^0 = \text{Im } d_*^{-1}$, and $\text{Im } \tilde{d}_*^0 = \text{Im } d_*^0 = \text{Ker } d_*^1 = \text{Hom}_{\mathcal{A}}(C, \text{Ker } d^1)$. ■

The natural functor $\mathcal{A} \rightarrow D_{\mathcal{C}}^b(\mathcal{A})$, which is the composition of the embedding $\mathcal{A} \rightarrow K^b(\mathcal{A})$ and the localization functor $K^b(\mathcal{A}) \rightarrow D_{\mathcal{C}}^b(\mathcal{A})$, is fully faithful. The proof is as [GZ, 2.9].

5.3. If \mathcal{A} has enough projective objects and $\mathcal{P} \subseteq \mathcal{C}$, then $K_{\mathcal{C}ac}^*(\mathcal{A})$ is a thick subcategory of $K_{ac}^*(\mathcal{A})$, where $K_{ac}^*(\mathcal{A})$ is the full subcategory of the homotopy category $K^*(\mathcal{A})$ consisting of acyclic complexes. By Lemma 2.1 there is a triangle-equivalence

$$D^*(\mathcal{A}) \cong D_{\mathcal{C}}^*(\mathcal{A}) / (K_{ac}^*(\mathcal{A}) / K_{\mathcal{C}ac}^*(\mathcal{A})),$$

and we have the localization functor $\pi_* : D_{\mathcal{C}}^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$. Note that π_* is an equivalence if and only if $\mathcal{C} = \mathcal{P}$.

Lemma 5.4. *Let \mathcal{C} be a full additive subcategory of abelian category \mathcal{A} . Then we have*

(i) ([CFH], Proposition 2.6) *A chain map $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ is a \mathcal{C} -quasi-isomorphism if and only if there are isomorphisms of abelian groups for any $C^\bullet \in K^-(\mathcal{C})$:*

$$\mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, f^\bullet[n]) : \mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, X^\bullet[n]) \cong \mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, Y^\bullet[n]), \quad \forall n \in \mathbb{Z}.$$

(ii) ([GZ], Lemma 2.2) *Let $C^\bullet \in K^-(\mathcal{C})$, and $f^\bullet : X^\bullet \longrightarrow C^\bullet$ be a \mathcal{C} -quasi-isomorphism. Then there is $g^\bullet : C^\bullet \longrightarrow X^\bullet$ such that $f^\bullet g^\bullet$ is homotopic to Id_{C^\bullet} .*

Thus, if in addition $X^\bullet \in K^-(\mathcal{C})$, then f^\bullet is a homotopy equivalence.

(iii) ([GZ], Proposition 2.8) *Let $C^\bullet \in K^-(\mathcal{C})$ and Y^\bullet be an arbitrary complex. Then $Q : f^\bullet \mapsto f^\bullet / \mathrm{Id}_{C^\bullet}$ gives an isomorphism $\mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, Y^\bullet) \cong \mathrm{Hom}_{D_{\mathcal{C}}(\mathcal{A})}(C^\bullet, Y^\bullet)$ of abelian groups.*

In particular, $K^b(\mathcal{C})$ can be viewed as a triangulated subcategory of $D_{\mathcal{C}}^b(\mathcal{A})$; and $K^-(\mathcal{C})$ can be viewed as a triangulated subcategory of $D_{\mathcal{C}}^-(\mathcal{A})$.

5.4. Let $K^{-,Cb}(\mathcal{C})$ denote the full subcategory of $K^-(\mathcal{C})$ given by

$$K^{-,Cb}(\mathcal{C}) := \{X^\bullet \in K^-(\mathcal{C}) \mid \exists N \in \mathbb{Z} \text{ such that } H^i \mathrm{Hom}_{\mathcal{A}}(C, X^\bullet) = 0, \\ \forall i \leq N, \forall C \in \mathcal{C}\}.$$

Then $K^{-,Cb}(\mathcal{C})$ is a thick triangulated subcategory of $K^-(\mathcal{C})$.

Lemma 5.5. *Let \mathcal{C} be a contravariantly finite subcategory of abelian category \mathcal{A} . Then for each $X^\bullet \in K^b(\mathcal{A})$ there is a \mathcal{C} -quasi-isomorphism $C_{X^\bullet} \longrightarrow X^\bullet$ with $C_{X^\bullet} \in K^{-,Cb}(\mathcal{C})$.*

Proof. The proof is similar as [GZ, Proposition 3.4]. Use induction on the width $w(X^\bullet)$, the number of i such that $X^i \neq 0$. Assume that $w(X^\bullet) = 1$. Then X^\bullet is the stalk complex of object X , say at degree 0. Since \mathcal{C} is contravariantly finite in \mathcal{A} , there exists a right \mathcal{C} -approximation $d^0 : C^0 \longrightarrow X$ of X . Taking a right \mathcal{C} -approximation $C^{-1} \longrightarrow \mathrm{Ker} d^0$, and continuing this process we get a complex

$$C^\bullet : \dots \longrightarrow C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} X \longrightarrow 0$$

with each $C^i \in \mathcal{C}$, such that $\mathrm{Hom}_{\mathcal{A}}(C, C^\bullet)$ is acyclic for each $C \in \mathcal{C}$. Put C_{X^\bullet} to be the complex obtained from C^\bullet by deleting X . By construction we get a \mathcal{C} -quasi-isomorphism $\phi_{X^\bullet} : C_{X^\bullet} \longrightarrow X^\bullet$ with $C_{X^\bullet} \in K^{-,Cb}(\mathcal{C})$.

Assume $w(X^\bullet) \geq 2$ with $X^j \neq 0$ and $X^i = 0$ for $i < j$. Then we have a distinguished triangle $X_1^\bullet \xrightarrow{u} X_2^\bullet \longrightarrow X^\bullet \longrightarrow X_1^\bullet[1]$ in $K^b(\mathcal{A})$, where $X_1^\bullet := X^j[-j-1]$ and X_2^\bullet is the

brutal truncated complex $X_{\geq j}^\bullet$. By induction there exist \mathcal{C} -quasi-isomorphisms

$$\phi_1 : C_{X_1^\bullet} \longrightarrow X_1^\bullet, \quad \phi_2 : C_{X_2^\bullet} \longrightarrow X_2^\bullet$$

with $C_{X_1^\bullet}, C_{X_2^\bullet} \in K^{-,cb}(\mathcal{C})$. By Lemma 5.4(i) ϕ_2 induces an isomorphism

$$\mathrm{Hom}_{K^-(\mathcal{A})}(C_{X_1^\bullet}, C_{X_2^\bullet}) \cong \mathrm{Hom}_{K^-(\mathcal{A})}(C_{X_1^\bullet}, X_2^\bullet).$$

Thus there is a unique chain map $f^\bullet : C_{X_1^\bullet} \longrightarrow C_{X_2^\bullet}$ such that $\phi_2 \circ f^\bullet = u \circ \phi_1$. Embedding f^\bullet into a distinguished triangle in $K^{-,cb}(\mathcal{C})$

$$C_{X_1^\bullet} \xrightarrow{f^\bullet} C_{X_2^\bullet} \longrightarrow C_{X^\bullet} \longrightarrow C_{X_1^\bullet}[1]$$

we get a unique complex C_{X^\bullet} in $K^{-,cb}(\mathcal{C})$. By the axiom of a triangulated category, there is $\phi_{X^\bullet} : C_{X^\bullet} \longrightarrow X^\bullet$ such that the diagram commutes

$$\begin{array}{ccccccc} C_{X_1^\bullet} & \xrightarrow{f^\bullet} & C_{X_2^\bullet} & \longrightarrow & C_{X^\bullet} & \longrightarrow & C_{X_1^\bullet}[1] \\ \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_{X^\bullet} \downarrow & & \phi_1[1] \downarrow \\ X_1^\bullet & \xrightarrow{u} & X_2^\bullet & \longrightarrow & X^\bullet & \longrightarrow & X_1^\bullet[1]. \end{array}$$

By using cohomological functors and the Five-Lemma it is easy to know that ϕ_{X^\bullet} is a \mathcal{C} -quasi-isomorphism. \blacksquare

The following result is due to J.Asadollahi, R.Hafezi, and R.Vahed [AHV, Theorem 3.3] (see also [GZ, Theorem 3.6] for the Gorenstein derived category). Since we need the equivalence $F : K^{-,cb}(\mathcal{C}) \longrightarrow D_{\mathcal{C}}^b(\mathcal{A})$ in its proof, and since the proof was omitted in [AVH], so we include a proof.

Proposition 5.6. ([AHV]) *Let \mathcal{C} be a contravariantly finite subcategory of abelian category \mathcal{A} . Then there is a triangle-equivalence $D_{\mathcal{C}}^b(\mathcal{A}) \cong K^{-,cb}(\mathcal{C})$, which fixes objects in $K^b(\mathcal{C})$.*

Proof. Let $F : K^{-,cb}(\mathcal{C}) \longrightarrow D_{\mathcal{C}}^-(\mathcal{A})$ be the composite of the embedding $K^{-,cb}(\mathcal{C}) \hookrightarrow K^-(\mathcal{A})$ and the localization functor $Q : K^-(\mathcal{A}) \longrightarrow D_{\mathcal{C}}^-(\mathcal{A})$. For each complex $X^\bullet \in K^{-,cb}(\mathcal{C})$, by definition there is an $N \in \mathbb{Z}$ such that $H^i \mathrm{Hom}_{\mathcal{A}}(C, X^\bullet) = 0$, $\forall i \leq N$, $\forall C \in \mathcal{C}$. Since the following chain map is a \mathcal{C} -quasi-isomorphism

$$\begin{array}{ccccccccccc} X^\bullet : & & \cdots & \longrightarrow & X^{N-2} & \longrightarrow & X^{N-1} & \longrightarrow & X^N & \longrightarrow & X^{N+1} & \longrightarrow \cdots \\ f^\bullet \downarrow & & & & \downarrow & & \downarrow & & \parallel & & \parallel & \\ \tau_{\geq N} X^\bullet : & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Ker} d^N & \longrightarrow & X^N & \longrightarrow & X^{N+1} & \longrightarrow \cdots \end{array}$$

it follows that there is an isomorphism $F(X^\bullet) \cong \tau_{\geq N} X^\bullet$ in $D_{\mathcal{C}}^-(\mathcal{A})$ with $\tau_{\geq N} X^\bullet \in D_{\mathcal{C}}^b(\mathcal{A})$. Thus the image of F falls in $D_{\mathcal{C}}^b(\mathcal{A})$, and hence F induces a triangle functor $K^{-,cb}(\mathcal{C}) \longrightarrow D_{\mathcal{C}}^b(\mathcal{A})$, again denoted by F (here we need to use Proposition 5.3). In particular, F fixes objects in $K^b(\mathcal{C})$, i.e., $F(X^\bullet) = X^\bullet$ for $X^\bullet \in K^b(\mathcal{C})$.

By Lemma 5.5 F is dense; and by Lemma 5.4(iii) F is fully faithful. \blacksquare

6. A relative description of bounded derived category

6.1. Let \mathcal{A} be an abelian category with enough projective objects, and \mathcal{C} a resolving subcategory of \mathcal{A} . By $K_{ac}^b(\mathcal{C})$ we denote the full subcategory of $K^-(\mathcal{C})$ consisting of those complexes which are homotopy equivalent to bounded acyclic complexes over \mathcal{C} . It is clear that $K_{ac}^b(\mathcal{C})$ is a triangulated subcategory of $K^-(\mathcal{C})$.

Lemma 6.1. *Let $C^\bullet \in K^{-,Cb}(\mathcal{C})$. If C^\bullet is acyclic, then $C^\bullet \in K_{ac}^b(\mathcal{C})$.*

Proof. Since $C^\bullet = (C^i, d^i)$ is upper bonded acyclic complex over \mathcal{C} , and \mathcal{C} is closed under kernels of epimorphisms, it follows that $\text{Im } d^i \in \mathcal{C}$, $\forall i \in \mathbb{Z}$. Since $C^\bullet \in K^{-,Cb}(\mathcal{C})$, by definition there exists an integer N such that $H^n \text{Hom}_{\mathcal{A}}(C, C^\bullet) = 0$, $\forall n \leq N$, $\forall C \in \mathcal{C}$. In particular $H^n \text{Hom}_{\mathcal{A}}(\text{Im } d^{n-1}, C^\bullet) = 0$. This implies that the induced epimorphism $\widetilde{d^{n-1}} : C^{n-1} \rightarrow \text{Im } d^{n-1}$ splits for $n \leq N$, and hence there is an isomorphism $C^\bullet \cong C'^\bullet$ in $K^-(\mathcal{C})$, where C'^\bullet is the complex

$$\cdots \rightarrow 0 \rightarrow \text{Im } d^{N-1} \hookrightarrow C^N \rightarrow C^{N+1} \rightarrow \cdots$$

with $C'^\bullet \in K_{ac}^b(\mathcal{C})$. Thus $C^\bullet \in K_{ac}^b(\mathcal{C})$. ■

6.2. The following result is another key step in proving Theorem 7.1, and also it seems to be of independent interest. If one takes \mathcal{C} to be \mathcal{P} , then it read as the well-known triangle equivalence $D^b(\mathcal{A}) \cong K^{-,b}(\mathcal{P})$. If $\mathcal{C} = \mathcal{GP}(\mathcal{A})$, then it is Theorem 5.1 of [KZ].

Theorem 6.2. *Let \mathcal{A} be an abelian category with enough projective objects, and \mathcal{C} a resolving contravariantly finite subcategory of \mathcal{A} . Then $K_{ac}^b(\mathcal{C})$ is a thick subcategory of $K^{-,Cb}(\mathcal{C})$, and we have a triangle-equivalence*

$$G : D^b(\mathcal{A}) \rightarrow K^{-,Cb}(\mathcal{C})/K_{ac}^b(\mathcal{C})$$

such that G sends an object $C \in K^b(\mathcal{C})$ to $C \in K^{-,Cb}(\mathcal{C})/K_{ac}^b(\mathcal{C})$.

Proof. Lemma 6.1 implies that $K_{ac}^b(\mathcal{C})$ is a thick subcategory of $K^{-,Cb}(\mathcal{C})$.

Let $F' : K_{ac}^b(\mathcal{C}) \rightarrow K_{ac}^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})$ be the composite of the embedding functor $K_{ac}^b(\mathcal{C}) \hookrightarrow K_{ac}^b(\mathcal{A})$ and the Verdier functor $Q : K_{ac}^b(\mathcal{A}) \rightarrow K_{ac}^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})$. We first claim that F' is a triangle equivalence.

Since $K_{ac}^b(\mathcal{A})$ is a triangulated subcategory of $K^b(\mathcal{A})$, it follows that $K_{ac}^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})$ is a triangulated subcategory of $K^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})$. By definition $K^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})$ is the \mathcal{C} -relative derived category $D_{\mathcal{C}}^b(\mathcal{A})$. By Lemma 5.4(iii) F' is fully faithful.

For each complex $X^\bullet \in K_{ac}^b(\mathcal{A})$, by Lemma 5.5 there is a \mathcal{C} -quasi-isomorphism $C^\bullet \rightarrow X^\bullet$ with $C^\bullet \in K^{-,Cb}(\mathcal{C})$. Since $\mathcal{C} \supseteq \mathcal{P}$, it follows that a \mathcal{C} -quasi-isomorphism is a quasi-isomorphism. Since X^\bullet is acyclic, it follows that C^\bullet is acyclic. By Lemma 6.1 $C^\bullet \in K_{ac}^b(\mathcal{C})$. By $X \cong F'(C^\bullet)$ in $K_{ac}^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})$ with $C^\bullet \in K_{ac}^b(\mathcal{C})$ we know that F' is dense. This proves the claim.

By construction F' is just the restriction of F to $K_{ac}^b(\mathcal{C})$, where F is the triangle-equivalence $K^{-,Cb}(\mathcal{C}) \rightarrow D_{\mathcal{C}}^b(\mathcal{A}) := K^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})$ given in the proof of Proposition

5.6. Hence we have a commutative diagram

$$\begin{array}{ccc} K_{ac}^b(\mathcal{C}) & \longrightarrow & K^{-,Cb}(\mathcal{C}) \\ \downarrow & & \downarrow \\ K_{ac}^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A}) & \longrightarrow & K^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A}) \end{array}$$

where the horizontal functors are embeddings, and the vertical ones are triangle-equivalences. Thus F induces a triangle-equivalence

$$K^{-,Cb}(\mathcal{C})/K_{ac}^b(\mathcal{C}) \cong (K^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})) / (K_{ac}^b(\mathcal{A})/K_{\mathcal{C}ac}^b(\mathcal{A})) \quad (*)$$

While by Lemma 2.1 we have a triangle-equivalence

$$\text{the right hand side of } (*) \cong K^b(\mathcal{A})/K_{ac}^b(\mathcal{A}) = D^b(\mathcal{A}).$$

This proves $D^b(\mathcal{A}) \cong K^{-,Cb}(\mathcal{C})/K_{ac}^b(\mathcal{C})$. Since this equivalence is induced by F , and F fixes objects in $K^b(\mathcal{C})$ by Proposition 5.6, it follows that it sends an object $C \in K^b(\mathcal{C})$ to $C \in K^{-,Cb}(\mathcal{C})/K_{ac}^b(\mathcal{C})$. \blacksquare

From the proof above, we see that the assumption “ \mathcal{C} is a resolving contravariantly finite subcategory” is used.

6.3. We need the following fact in the next section.

Proposition 6.3. *Let \mathcal{A} be an abelian category with enough projective objects, and \mathcal{C} a full additive subcategory of \mathcal{A} with $\mathcal{P} \subseteq \mathcal{C}$. Then there is a functorial isomorphism of abelian groups for each $P \in K^-(\mathcal{P})$ and $C \in K^-(\mathcal{C})$*

$$\text{Hom}_{K^-(\mathcal{C})}(P, C) \cong \text{Hom}_{K^-(\mathcal{C})/K_{ac}^b(\mathcal{C})}(P, C)$$

given by $f \mapsto f/\text{Id}_P$, $\forall f \in \text{Hom}_{K^-(\mathcal{C})}(P, C)$.

Proof. The proof is similar as in the case of derived category. Since this assertion will be used, for the completeness we include a justification.

Recall a well-known fact: if $t : Z \rightarrow P$ is a quasi-isomorphism with $P \in K^-(\mathcal{P})$, then there is $g : P \rightarrow Z$ such that tg is homotopic to Id_P (cf. Lemma 5.4(ii)).

Now assume $f/\text{Id}_P = 0$. By definition we have a commutative diagram in $K^-(\mathcal{C})$

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \uparrow t & \searrow f & \\ P & \xleftarrow{t} & Z & \xrightarrow{0} & C \\ & \swarrow & \downarrow t & \searrow 0 & \\ & & P & & \end{array}$$

where $t : Z \rightarrow P$ a chain map such that $\text{Con}(t) \in K_{ac}^b(\mathcal{C})$. Thus t is a quasi-isomorphism, and hence there is $g : P \rightarrow Z$ such that tg is homotopic to Id_P . Thus by $ft = 0$ we have $f = f(tg) = 0$.

Assume $f/s \in \text{Hom}_{K^-(\mathcal{C})/K_{ac}^b(\mathcal{C})}(P, C)$, where $s : Z \rightarrow P$ with $\text{Con}(s) \in K_{ac}^b(\mathcal{C})$, and $f : Z \rightarrow C$. Since s is quasi-isomorphism, there is $g : P \rightarrow Z$ such that sg is homotopic

to Id_P , and hence we get a commutative diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow s & \uparrow g & \searrow f & \\
 P & \xleftarrow{=} & P & \xrightarrow{fg} & C \\
 & \swarrow = & \downarrow = & \nearrow fg & \\
 & & P & &
 \end{array}$$

This means $f/s = fg/\text{Id}_P$. ■

6.4. For later use, we need to investigate $K^-(\mathcal{C})/K_{ac}^b(\mathcal{C})$ in more details.

Let \mathcal{A} be an abelian category with enough projective objects, and \mathcal{C} a resolving subcategory of \mathcal{A} . An object $I \in \mathcal{C}$ is a (relative) *injective object* of \mathcal{C} , provided that the functor $\text{Hom}_{\mathcal{A}}(-, I)$ sends any short exact sequence $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$ with $X_i \in \mathcal{C}$, $i = 1, 2, 3$, to an exact sequence. Clearly, I is an injective object of \mathcal{C} if and only if $\text{Ext}_{\mathcal{A}}^1(X, I) = 0$ for each $X \in \mathcal{C}$, also if and only if $\text{Ext}_{\mathcal{A}}^i(X, I) = 0$ for each $X \in \mathcal{C}$ and for $i \geq 1$,

Lemma 6.4. *Let \mathcal{C} be a resolving subcategory of \mathcal{A} , and $G = (G^i, d_G^i) \in K_{ac}^-(\mathcal{C})$. Assume that $I = (I^i, d_I^i)$ is a bounded complex such that all I^i are injective objects of \mathcal{C} . Then $\text{Hom}_{K^-(\mathcal{A})}(G, I) = 0$.*

Proof. The proof is similar to the case of $\mathcal{C} = \mathcal{A}$, which is well-known. For the completeness we include a proof.

Let $f : G \rightarrow I$ be a chain map. We need to show that f is null-homotopic. We construct a homotopy $s = (s^i)$ by induction. Assume that we have constructed $s^i : G^i \rightarrow I^i$ for $i \leq m$, such that $f^{i-1} = d_I^{i-2}s^{i-1} + s^i d_G^{i-1}$ for $i \leq m$. Since G is an upper bounded acyclic complex with all $G^i \in \mathcal{C}$, and \mathcal{C} is closed under the kernels of epimorphisms, it follows that $\text{Im} d_G^j \in \mathcal{C}$, $\forall j \in \mathbb{Z}$. Since

$$(f^m - d_I^{m-1}s^m)d_G^{m-1} = 0$$

it follows that $f^m - d_I^{m-1}s^m$ factors through $\text{Coker} d_G^{m-1} = \text{Im} d_G^m$. Since I^m is an injective object of \mathcal{C} , it follows that there is $s^{m+1} : G^{m+1} \rightarrow I^m$ such that

$$f^m - d_I^{m-1}s^m = s^{m+1}d_G^m.$$

This completes the proof. ■

Lemma 6.5. *Let \mathcal{C} be a resolving subcategory of \mathcal{A} , and $C \in K^-(\mathcal{C})$. Assume that I is a bounded complex such that all I^i are injective objects of \mathcal{C} . If $t : I \rightarrow C$ a quasi-isomorphism, then there exists a chain map $s : C \rightarrow I$ such that $st = \text{Id}_I$ in $K^-(\mathcal{A})$.*

Proof. By Lemma 6.4 $\text{Hom}_{K^-(\mathcal{A})}(\text{Con}(t), I) = 0 = \text{Hom}_{K^-(\mathcal{A})}(\text{Con}(t)[-1], I)$. Applying $\text{Hom}_{K^-(\mathcal{A})}(-, I)$ to the distinguished triangle $I \xrightarrow{t} C \rightarrow \text{Con}(t) \rightarrow I[1]$ we see that $\text{Hom}_{K^-(\mathcal{A})}(C, I) \xrightarrow{\text{Hom}(t, I)} \text{Hom}_{K^-(\mathcal{A})}(I, I)$ is an isomorphism, from which the assertion follows. ■

Proposition 6.6. *Let \mathcal{A} be an abelian category with enough projective objects, and \mathcal{C} a resolving subcategory of \mathcal{A} . Assume that I is a bounded complex such that all I^i are*

injective objects of \mathcal{C} . Then for each $C \in K^-(\mathcal{C})$ there is a functorial (in C and in I) isomorphism

$$\mathrm{Hom}_{K^-(\mathcal{C})}(C, I) \cong \mathrm{Hom}_{K^-(\mathcal{C})/K_{ac}^b(\mathcal{C})}(C, I).$$

Proof. Here we need to use the left fraction construction of $K^-(\mathcal{C})/K_{ac}^b(\mathcal{C})$. The isomorphism is given by $f \mapsto \mathrm{Id}_I \backslash f$, $\forall f \in \mathrm{Hom}_{K^-(\mathcal{C})}(C, I)$. The proof is dual to the one of Proposition 6.3, by using Lemma 6.5. We omit the details. \blacksquare

7. Main results

7.1. Now we are in position to prove

Theorem 7.1. *Let A be an Artin algebra with $\mathrm{gl.dim} A = \infty$. Assume that there are modules T and M in $A\text{-mod}$ with $\mathrm{inj.dim} T < \infty$, such that ${}^\perp T = \mathrm{add} M$. Then $D^b(A\text{-mod})$ admits a categorical resolution $D^b(B\text{-mod})$ with $B = (\mathrm{End}_A M)^{op}$.*

Proof. By Theorem 4.3 $\mathrm{gl.dim} B < \infty$, i.e., $D^b(B\text{-mod})$ is smooth.

The equivalence $\mathrm{Hom}_A(M, -) : \mathrm{add} M \rightarrow \mathcal{P}(B\text{-mod})$ of categories induces pointwisely a triangle-equivalence $K^{-, \mathrm{add} M} b(\mathrm{add} M) \cong K^{-, b}(\mathcal{P}(B\text{-mod}))$. Since $D^b(B\text{-mod}) \cong K^{-, b}(\mathcal{P}(B\text{-mod}))$, we have a triangle-equivalence

$$F : D^b(B\text{-mod}) \cong K^{-, \mathrm{add} M} b(\mathrm{add} M).$$

Since $\mathrm{add} M = {}^\perp T$, it follows that $\mathrm{add} M$ is a resolving contravariantly finite subcategory of $A\text{-mod}$, and hence by Theorem 6.2 we have a triangle-equivalence

$$G : D^b(A\text{-mod}) \rightarrow K^{-, \mathrm{add} M} b(\mathrm{add} M)/K_{ac}^b(\mathrm{add} M)$$

such that G sends an object $P \in K^b(\mathcal{P}(A\text{-mod}))$ to $P \in K^{-, \mathrm{add} M} b(\mathrm{add} M)/K_{ac}^b(\mathrm{add} M)$, i.e., $GP = P$. Thus, we get a triangle functor

$$\pi_* := G^{-1}VF : D^b(B\text{-mod}) \rightarrow D^b(A\text{-mod}),$$

where $V : K^{-, \mathrm{add} M} b(\mathrm{add} M) \rightarrow K^{-, \mathrm{add} M} b(\mathrm{add} M)/K_{ac}^b(\mathrm{add} M)$ is the Verdier functor.

On the other hand, by Proposition 3.2 $D^b(A\text{-mod})^{\mathrm{perf}} = K^b(\mathcal{P}(A\text{-mod}))$. Thus we have a triangle functor

$$\pi^* := F^{-1}\sigma : D^b(A\text{-mod})^{\mathrm{perf}} \rightarrow D^b(B\text{-mod})$$

where σ is the embedding $\sigma : K^b(\mathcal{P}(A\text{-mod})) \hookrightarrow K^{-, \mathrm{add} M} b(\mathrm{add} M)$.

The diagram

$$\begin{array}{ccc} D^b(B\text{-mod}) & \xrightarrow{\pi_*} & D^b(A\text{-mod}) \\ \downarrow F & & \downarrow G \\ K^{-, \mathrm{add} M} b(\mathrm{add} M) & \xrightarrow{V} & K^{-, \mathrm{add} M} b(\mathrm{add} M)/K_{ac}^b(\mathrm{add} M) \end{array}$$

commutes. Since $K_{ac}^b(\mathrm{add} M)$ is thick in $K^{-, \mathrm{add} M} b(\mathcal{C})$ (cf. Theorem 6.2), we have $\mathrm{Ker} V = K_{ac}^b(\mathrm{add} M)$. It follows that $\mathrm{Ker} \pi_* = F^{-1}(K_{ac}^b(\mathrm{add} M))$, and π_* induces a triangle-equivalence $D^b(B\text{-mod})/\mathrm{Ker} \pi_* \cong D^b(A\text{-mod})$.

Notice that π^* is left adjoint to π_* on $K^b(\mathcal{P}(A\text{-mod}))$. In fact, for $P \in K^b(\mathcal{P}(A\text{-mod}))$ and $X \in D^b(B\text{-mod})$ we have

$$\begin{aligned} \text{Hom}_{D^b(B\text{-mod})}(\pi^*P, X) &\cong \text{Hom}_{K^-, \text{addM } b(\text{addM})}(\sigma P, FX) \\ &\cong \text{Hom}_{K^-, \text{addM } b(\text{addM})}(P, FX); \end{aligned}$$

and

$$\begin{aligned} \text{Hom}_{D^b(A\text{-mod})}(P, \pi_*X) &\cong \text{Hom}_{K^-, \text{addM } b(\text{addM})/K_{ac}^b(\text{addM})}(GP, VFX) \\ &\cong \text{Hom}_{K^-, \text{addM } b(\text{addM})/K_{ac}^b(\text{addM})}(P, FX) \end{aligned}$$

(note that $GP = P$ and $VFX = FX$). So, it suffices to prove that there is a functorial isomorphism

$$\zeta_{P, FX} : \text{Hom}_{K^-, \text{addM } b(\text{addM})}(P, FX) \cong \text{Hom}_{K^-, \text{addM } b(\text{addM})/K_{ac}^b(\text{addM})}(P, FX).$$

This follows from Proposition 6.3 by taking $\mathcal{C} = \text{addM}$.

Finally, saying that the unit $\text{Id}_{\mathcal{D}^{\text{perf}}} \longrightarrow \pi_*\pi^* = G^{-1}V\sigma$ is a natural isomorphism of functors amounts to saying that

$$\zeta_P = \zeta_{P, P}(\text{Id}_P) = \text{Id}_P / \text{Id}_P : P \longrightarrow P$$

is an isomorphism in $K^-, \text{addM } b(\text{addM})/K_{ac}^b(\text{addM})$ for each $P \in K^b(\mathcal{P}(A\text{-mod}))$. This is trivially true.

All together triple $(D^b(B\text{-mod}), \pi_*, \pi^*)$ is a categorical resolution of $D^b(A\text{-mod})$. \blacksquare

7.2. Theorem 7.1 is stated for the finitely generated module category $A\text{-mod}$. By the similar argument with a minor change we can prove its version for $A\text{-Mod}$. For the contravariantly finiteness of AddM in $A\text{-Mod}$, we need the assumption that “ M is finitely generated” (cf. Example 2.5(i)). For $T \in A\text{-mod}$, let ${}^{\perp \text{big}}(\text{Add}T)$ denote the full subcategory of $A\text{-Mod}$ given by ${}^{\perp \text{big}}(\text{Add}T) = \{X \in A\text{-Mod} \mid \text{Ext}_A^i(X, T') = 0, \forall i \geq 1, \forall T' \in \text{Add}T\}$ (here “big” refers to work in $A\text{-Mod}$). Note that there is no a module $T' \in A\text{-Mod}$ such that ${}^{\perp \text{big}}(\text{Add}T) = {}^{\perp \text{big}}T'$.

Theorem 7.2. *Let A be an Artin algebra with $\text{gl.dim}A = \infty$. Assume that there are modules T and M in $A\text{-mod}$ with $\text{inj.dim}T < \infty$, such that ${}^{\perp \text{big}}(\text{Add}T) = \text{Add}M$. Then $D^b(A\text{-Mod})$ admits a categorical resolution $D^b(B\text{-Mod})$ with $B = (\text{End}_A M)^{\text{op}}$.*

Proof. First, the condition ${}^{\perp \text{big}}(\text{Add}T) = \text{Add}M$ implies ${}^{\perp}T = \text{add}M$. The argument is as follows:

$${}^{\perp}T = {}^{\perp \text{big}}(\text{Add}T) \cap A\text{-mod} = \text{Add}M \cap A\text{-mod} = \text{add}M.$$

By Theorem 4.3 $\text{gl.dim}B < \infty$, i.e., $D^b(B\text{-Mod})$ is smooth. Since M is finitely generated, $\text{Hom}_A(M, -) : \text{AddM} \longrightarrow \mathcal{P}(B\text{-Mod})$ is again an equivalence of categories. Since

$$\text{Hom}_A(X, M') \cong \text{Hom}_B(\text{Hom}_A(M, X), \text{Hom}_A(M, M')), \forall X \in \text{AddM}, \forall M' \in \text{AddM},$$

it follows that this equivalence induces pointwisely a triangle-equivalence $K^-, \text{AddM } b(\text{AddM}) \cong K^-, {}^{\perp b}(\mathcal{P}(B\text{-Mod}))$, and hence we get a triangle-equivalence

$$F : D^b(B\text{-Mod}) \cong K^-, \text{AddM } b(\text{AddM}).$$

Since $\text{Add}M = {}^{\perp_{\text{big}}}(\text{Add}T)$, it follows that $\text{Add}M$ is a resolving subcategory of $A\text{-Mod}$. Also, $\text{Add}M$ is contravariantly finite in $A\text{-Mod}$ by Example 2.5(i).

The rest of the proof is similar with the one for Theorem 7.1, just replacing $\text{add}M$ by $\text{Add}M$, $A\text{-mod}$ by $A\text{-Mod}$, and $B\text{-mod}$ by $B\text{-Mod}$. We omit the details. \blacksquare

7.3. Let us see some special cases of Theorems 7.1 and 7.2. We have a reformulation of the Auslander algebra:

Corollary 7.3. *Let A be a representation-finite Artin algebra with $\text{gl.dim}A = \infty$, and B its Auslander algebra. Then*

- (i) $D^b(B\text{-mod})$ is a categorical resolution $D^b(A\text{-mod})$.
- (ii) $D^b(B\text{-Mod})$ is a categorical resolution $D^b(A\text{-Mod})$.

Proof. Put T to be an injective module in $A\text{-mod}$, and M to be the direct sum of all the pairwise non-isomorphic finitely generated indecomposable modules.

By Theorem 7.1 we get (i).

Since A is representation-finite, any A -module is a direct sum of finitely generated indecomposable modules (see [A2]). It follows that ${}^{\perp_{\text{big}}}T = A\text{-Mod} = \text{Add}M$. By Theorem 7.2 we get (ii). \blacksquare

7.4. A module $T \in A\text{-mod}$ is a *cotilting module* ([AR]), if

- (i) $\text{inj.dim}T \leq 1$;
- (ii) $\text{Ext}_A^1(T, T) = 0$; and
- (iii) There is an exact sequence $0 \rightarrow T_0 \rightarrow T_1 \rightarrow D(A_A) \rightarrow 0$ with $T_i \in \text{add}T$, $i = 0, 1$.

An module $X \in A\text{-mod}$ is *cogenerated by T* , if X can be embedded as an A -module into a finite direct sum of copies of T . Then X is cogenerated by a cotilting module T if and only if $X \in {}^{\perp}T$ ([HR]). By Theorem 7.1 we have

Corollary 7.4. *Let A be an Artin algebra with $\text{gl.dim}A = \infty$. Assume that A has a cotilting module T such that there are only finitely many pairwise non-isomorphic indecomposable A -modules which are cogenerated by T . Then $D^b(A\text{-mod})$ admits a categorical resolution.*

7.5. Finally, we consider CM-finite Gorenstein algebras.

Theorem 7.5. *Let A be a CM-finite Gorenstein algebra with $\text{gl.dim}A = \infty$, and B its relative Auslander algebra. Then*

- (i) $D^b(B\text{-mod})$ is a weakly crepant categorical resolution $D^b(A\text{-mod})$.
- (ii) $D^b(B\text{-Mod})$ is a weakly crepant categorical resolution $D^b(A\text{-Mod})$.

Proof. Take $T = {}_AA$, and M to be the direct sum of all the pairwise non-isomorphic finitely generated indecomposable Gorenstein-projective modules, in Theorem 7.1 and 7.2.

(i) Since A is CM-finite, we have $M \in A\text{-mod}$ and $\mathcal{GP}(A\text{-mod}) = \text{add}M$. Since A is Gorenstein, it follows from [EJ, Corollary 11.5.3] that $\mathcal{GP}(A\text{-mod}) = {}^{\perp}({}_AA)$. Thus ${}^{\perp}T =$

addM . Then $D^b(A\text{-mod})$ has a categorical resolution $(D^b(B\text{-mod}), \pi_*, \pi^*)$ by Theorem 7.1. It remains to see that π^* is right adjoint to π_* on $K^b(\mathcal{P}(A\text{-mod}))$. As in the proof of Theorem 7.1 it suffices to prove that there is a functorial isomorphism

$$\text{Hom}_{K^-, \text{addM}} {}^b(\text{addM})(FX, P) \cong \text{Hom}_{K^-, \text{addM}} {}^b(\text{addM})/K_{ac}^b(\text{addM})(FX, P).$$

This follows from Proposition 6.6 by taking $\mathcal{C} = \text{addM} = \mathcal{GP}(A\text{-mod})$, since projective modules are injective objects of $\mathcal{GP}(A\text{-mod})$.

(ii) Since A is a CM-finite Gorenstein algebra, any Gorenstein-projective A -module is a direct sum of finitely generated indecomposable Gorenstein-projective modules (see [C1]). It follows that $\mathcal{GP}(A\text{-Mod}) = \text{AddM}$. Since A is Gorenstein, it follows from [EJ, Corollary 11.5.3] (or [Bel1, Proposition 3.10]) that $\mathcal{GP}(A\text{-Mod}) = {}^{\perp_{\text{big}}}(\text{Add } {}_A A)$. Thus ${}^{\perp_{\text{big}}}(\text{Add } T) = \text{AddM}$. Then $D^b(B\text{-Mod})$ is a categorical resolution of $D^b(A\text{-Mod})$ by Theorem 7.2. By the similar argument as in (i) we know that it is weakly crepant. ■

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